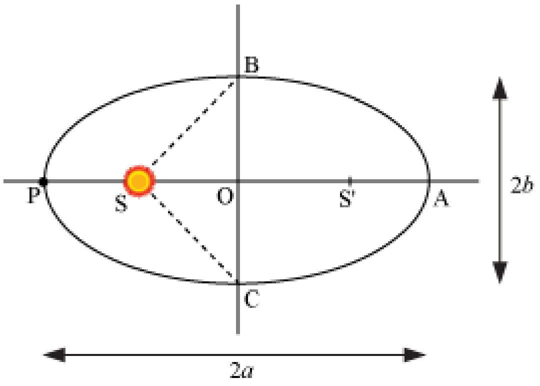
# Gravitation

## Kepler's Laws

### Kepler’s First Law (Law of Orbits)

- Every planet revolves around the sun in an elliptical orbit. The sun is situated at one foci of the ellipse.




- The closest point is P, called the perihelion, and the farthest point is A, called the aphelion.

### Kepler’s Second Law (Law of Areas)

The line joining a planet to the sun sweeps out equal areas in equal intervals of time, i.e., the areal velocity of the planet around the sun is constant.

**The law of areas can be understood as a consequence of the conservation of angular momentum.**

The geometrical interpretation of the angular momentum of a point mass body of mass, *m* rotating about a given axis is,

$$\vec{L} = 2m \times \frac{d\vec{A}}{dt} \qquad ...(i)$$

A planet revolves around the sun under the influence of the gravitational pull of the sun.

Thus, gravitational pull $(\vec{F})$ is a centred force acting towards the sun.

∴ Torque, $\vec{\tau} = \vec{r} \times \vec{F} = rF \sin 180 \hat{n} = 0$

[∵ Angle between $\vec{r}$ and $\vec{F}$ is 180°]

As $\vec{\tau} = \frac{d\vec{L}}{dt} = 0;$

So $\vec{L}$ = Constant ...(ii)

From (i) and (ii), we get

$2m \times \frac{d\vec{A}}{dt} =$ Constant

$\Rightarrow \frac{d\vec{A}}{dt}$ = Constant

This means that the areal velocity of a planet revolving around the sun is constant.

**Kepler's Third Law (Law of Periods)**

The square of the time period of revolution of a planet around the sun is directly proportional to the cube of the semi major axis of its elliptical orbit, i.e., $T^2 \alpha R^3$

**Universal Law of Gravitation and Gravitational Constant**

- Consider two bodies of masses $m_1$ and $m_2$ with their centres separated by a distance $r$.


Let $F$ be the force of gravitational attraction between the two bodies. According to Newton's law of gravitation,

$F \alpha m_1 m_2$

And, $F \alpha \frac{1}{r^2}$

Combining both the factors,

$$F \alpha \frac{m_1 m_2}{r^2}$$

$$\therefore F = G\frac{m_1 m_2}{r^2} \quad \text{...(i)}$$

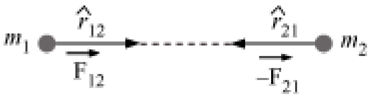
Where, $G$ is constant of proportionality known as gravitational constant

Its S.I value is $6.67 \times 10^{-11}$ $Nm^2\ kg^{-2}$.

- If $m_1 = m_2 = 1$ units and $r = 1$ unit, then from equation (i), $F = G$

  Therefore, the universal gravitational constant ($G$) is numerically equal to the force of attraction between two bodies for unit masses, separated by unit distance.

- **Some special features of gravitational force**
- It does not depend on the nature of the medium in which the masses are placed.
- It is extremely small in case of light bodies whereas it becomes appreciable in case of heavy bodies.
- It is a conservative force.
- It is always attractive.
- It is a central force.
- **Vector form of Newton's Law of Gravitation**




Consider two particles of masses $m_1$ and $m_2$.

Let $\vec{r}_{12}$ = Displacement vector from $m_1$ to $m_2$

$\vec{r}_{21}$ = Displacement vector from $m_2$ to $m_1$

$\vec{F}_{21}$ = Gravitational force exerted by $m_1$ on $m_2$

$\vec{F}_{12}$ **=** Gravitational force exerted by $m_2$ on $m_1$

$\hat{r}_{12}$ = Unit vector pointing towards $m_2$

$\hat{r}_{21}$ = Unit vector pointing towards $m_1$

In vector form, the Newton's law of gravitation is written as

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{21} \qquad \text{...(i)}$$

Negative sign indicates that the direction of $\overrightarrow{F_{12}}$ is opposite to that of $\hat{r}_{21}$.

Similarly, $\vec{F}_{21} = -G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{12}$

Where, $\hat{r}_{21}$ is a unit vector pointing towards $m_2$

Also, $\hat{r}_{21} = -\hat{r}_{12}$ and $r_{21}^2 = r_{12}^2$

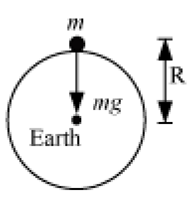
$$\therefore \vec{F}_{21} = G\frac{m_1 m_2}{r_{21}^2}\hat{r}_{21} \qquad \text{...(2)}$$

From equations (1) and (2),

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

**Acceleration Due To Gravity of Earth**

- When an object is dropped from a height, it falls towards the earth. Thus, the acceleration produced in an object due to the force of gravity is known as acceleration due to gravity (g).
- **Acceleration due to gravity at the surface of earth**



$M$ – Mass of earth

$m$ – Mass of body lying above the surface of earth

$R$ – Radius of earth

According to Newton's law of gravitation, the force of attraction between the body and the earth is

$$F = \frac{GMm}{R^2} \quad \dots \text{(i)}$$

However, the force with which a body is attracted towards the earth gives the weight of the body.

$\therefore F = mg$ … (ii)

Equating (i) and (ii),

$$mg = G\frac{Mm}{R^2}$$

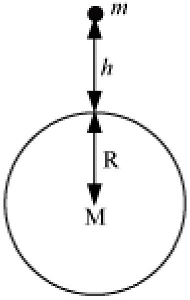
$$\boxed{\therefore g = \frac{GM}{R^2}}$$

**Acceleration due to gravity below and above the surface of earth**

- **Variation of g with altitude (height)**

Let us consider a body of mass $m$ lying on the surface of the earth of mass $M_E$ and radius $R_E$. The value of acceleration due to gravity on the surface of earth is given by,

$$g = \frac{GM_E}{R_E^2} \quad \dots \text{(i)}$$



Let '*h*' be the height of the body of mass '*m*' from the surface of earth where the value of acceleration due to gravity is $g_h$.

Then, $g_h = \frac{GM_E}{(R_E + h)^2}$ ...(ii)

[$\because (R_E + h)$ is the distance between the centres of body and earth]

Dividing (ii) by (i), we obtain

$$\frac{g_h}{g} = \frac{R_E^2}{(R_E + h)^2}$$

$$\frac{g_h}{g} = \frac{R_E^2}{\left[R_E\left(1 + \frac{h}{R_E}\right)\right]^2} = \frac{R_E^2}{R_E^2\left(1 + \frac{h}{R_E}\right)^2}$$

$$\text{or } \frac{g_h}{g} = \frac{1}{\left(1 + \frac{h}{R_E}\right)^2} = \left(1 + \frac{h}{R_E}\right)^{-2}$$

Since $h << R_E$,

$\frac{h}{R_E}$ is very small as compared to 1

Applying binomial theorem and neglecting squares and higher powers of $\frac{h}{R_E}$, we obtain

$$\frac{g_h}{g} = 1 - \frac{2h}{R_E}$$

$$g_h = g\left(1 - \frac{2h}{R_E}\right)$$

$$\therefore g_h = g - \frac{2h}{R_E}g$$

It is evident from the formula that the value of acceleration due to gravity decreases with increase in height above the surface of earth.

- **Variation of 'g' with depth**

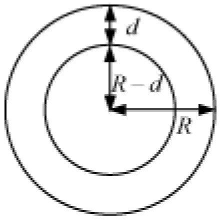
Assume that the earth is a homogeneous sphere of radius $R_E$ and mass $M_E$. The value of 'g' on the surface of earth is given by,

$$g = \frac{GM_E}{R_E^2}$$

Where, $\rho$ is the mean density of earth

$$g = \frac{4\pi R_E^3 \rho G}{3R_E^2} = \frac{4}{3}\pi R_E \rho G \quad \ldots(i)$$

If the body be taken to a depth '*d*' below the free surface of earth where the value of acceleration due to gravity is $g_d$, then




Here, the force of gravity acting on the body is given by,

$$g_d = \frac{GM'}{(R_E - d)^2}$$

Where, $M'$ is the mass of inner solid of radius $(R - d)$

$$g_d = \frac{G}{(R_E - d)^2} \times \frac{4}{3}\pi(R-d)^3 \times \rho$$

$$g_d = \frac{4}{3}\pi G(R_E - d)\rho \quad \ldots (ii)$$

[$\because M' = \frac{4}{3}\pi(R-d)^3\rho$, since mass = volume × density]

Dividing (ii) by (i), we obtain

$$\frac{g_d}{g} = \frac{\frac{4}{3}\pi G(R_E - d)\rho}{\frac{4}{3}\pi G R_E \rho} = \frac{R_E - d}{R_E}$$

$$\frac{g_d}{g} = 1 - \frac{d}{R_E}$$

$$\Rightarrow \boxed{g_d = g\left(1 - \frac{d}{R_E}\right)}$$

Thus, it is evident from the formula that the value of 'g' decreases with increase in depth.

**Gravitational Potential Energy and Escape Speed**

**Gravitational Potential**

- Gravitational potential at a point in a gravitational field is defined as the amount of work done in bringing a body of unit mass from infinity to that point without acceleration.

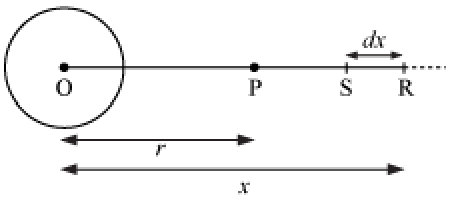
∴ Gravitational potential at any point $= \frac{W}{m_0}$

Where, $W$ is the amount of work done in bringing a body of mass $m_0$ from infinity to that point

- Expression for gravitational potential at a point

Assume earth to be a perfect sphere of radius $R_E$ and mass $M_E$. We have to find gravitational potential at any point P distant '$r$' from the centre of the earth ($r > R_E$).

Take two points R and S on the line joining OP.




The gravitational force of attraction on a body of unit mass at point '$R$' is

$$F = \frac{GM_E \times 1}{x^2}$$

$$F = \frac{GM_E}{x^2}$$

Small amount of work done in bringing the body of unit mass from *R* to *S* through a small distance *RS* (= *dx*) is

$$dW = Fdx$$

$$dW = \frac{GM_E}{x^2} dx$$

Total work done in bringing the body from infinity to point P is

$$W = \int_{\infty}^{r} \frac{GM_E}{x^2} dx = -\left(\frac{GM_E}{x}\right)_{\infty}^{r}$$

Or, $W = -GM_E\left(\frac{1}{r} - \frac{1}{\infty}\right) = -\frac{GM_E}{r}$

This work done is the measure of gravitational potential at *P* (that is *V*).

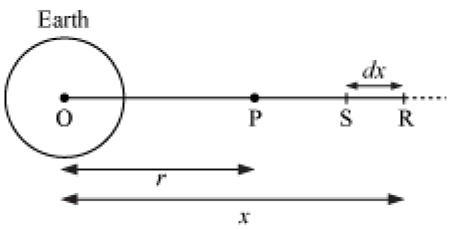
$$\therefore \boxed{V = W = -\frac{GM_E}{r}} \qquad \text{...(i)}$$

**Gravitational Potential Energy**

- Gravitational potential energy of a body at a point in a gravitational field of another body is defined as the amount of work done in bringing the given body from infinity to that point without acceleration.

- Expression for gravitational potential energy:

  Let $M_E$ be the mass of earth and its radius be $R_E$. We have to find the gravitational potential energy of the body of mass *m* placed at point P in the gravitational field.



The gravitational force on the body at point *R* will be

$$F = \frac{GM_E m}{x^2}$$

Small amount of work done in bringing the body from *R* to *S* through a small distance '*dx*' is

*dW* = *Fdx*

$$dW = \frac{GM_E m}{x^2} dx$$

Total work done in bringing the body from infinity to point P is given by,

$$W = \int_{\infty}^{r} \frac{GM_E m}{x^2} dx = GM_E m \int_{\infty}^{r} x^{-2} dx$$

Or, $W = -GM_E m \left[ \frac{1}{x} \right]_{\infty}^{r}$

Or, $W = -GM_E m \left[ \frac{1}{r} - \frac{1}{\infty} \right]$

Or, $W = -\frac{GM_E m}{r}$

This work done is stored in the body as gravitational potential energy *U*.

$$U = -\frac{GM_E m}{r}$$

However, gravitational potential,

$$V = -\frac{GM_E}{r}$$

$$\therefore \boxed{U = V \times m}$$

i.e.,

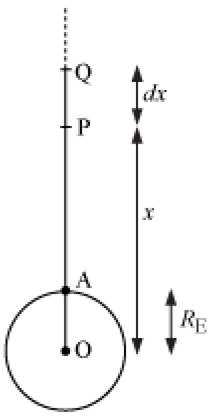
Gravitational potential energy = Gravitational potential × Mass of the body

**Escape Velocity**

- Escape velocity is defined to be the minimum velocity that an object must have in order to escape the gravitational field of the earth (i.e., escape the earth without ever falling back).

- Expression for escape velocity:

  Assume earth to be a perfect sphere of mass $M_E$ and radius $R_E$. A body of mass '$m$' is projected from point 'A' on the surface of earth. Take two points, P and Q, on the line joining O and A.




Gravitational force of attraction on the body at point P is

$$F = \frac{GM_E m}{x^2} dx$$

Small amount of work done in bringing the body from P to Q against gravitational attraction,

$$dW = Fdx = \frac{GM_E m}{x^2} dx$$

Total work done in taking the body against gravitational attraction from the surface of earth to a region beyond the gravitational field of earth (i.e, from $x = R_E$ to $x = \infty$) is given by,

$$W = \int_{R_E}^{\infty} \frac{GM_E m}{x^2} dx$$

$$W = GM_E m \int_{R_E}^{\infty} x^{-2} dx$$

$$W = -GM_E m \left[ \frac{1}{x} \right]_{R_E}^{\infty}$$

$$W = -GM_E m \left[ \frac{1}{\infty} - \frac{1}{R_E} \right]$$

$$W = \frac{GM_E m}{R_E}$$

This work done is at the cost of kinetic energy given to the body at the surface of the earth.

Kinetic energy of the body $= \frac{1}{2} m v_e^2$

Where, $v_e$ is the escape velocity of the body projected

$$\therefore \frac{1}{2} m v_e^2 = \frac{GM_E m}{R_E}$$

$$v_e^2 = \frac{2GM_E}{R_E}$$

$$v_e = \sqrt{\frac{2GM_E}{R_E}}$$

As $g = \frac{GM_E}{R_E^2}$ ($\therefore GM_E = gR_E^2$),

$$\therefore \boxed{v_e = \sqrt{2gR_E}}$$

For earth, g = 9.8 $ms^{-2}$, $R_E = 6.4 \times 10^6$ m

$$\therefore v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$
$$= 11.2 \times 10^3 \, ms^{-1}$$

$$\boxed{v_e = 11.2 \text{ kms}^{-1}}$$

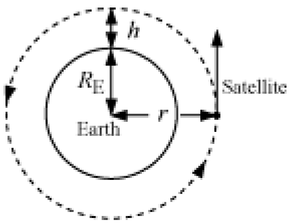
**Earth's Satellites and Energy of an Orbiting Satellite**

**Satellite**

- A heavenly object revolving around another much larger body is called a satellite. Example – Earth has a natural satellite – moon

- **Orbital speed and time period of satellite**

- Orbital speed – Speed required to put the satellite into a given orbit around earth




Let

$M_E$ = Mass of earth

$R_E$ = Radius of earth

$h$ = Height of the satellite

$v$ = Orbital speed of the satellite

$m$ = Mass of the satellite

$r = R_E + h$ = Radius of the orbit of the satellite

The satellite revolves around the planet in a circular orbit; the gravitational pull provides the required centripetal force to the satellite.

$$\therefore \frac{mv^2}{r} = \frac{GM_E m}{r^2}$$

$$v = \sqrt{\frac{GM_E}{r}}$$

The value of g on the surface of earth is given by,

$$g = \frac{GM_E}{R_E^{\ 2}}$$

$$gR_E^{\ 2} = GM_E$$

$$\therefore v = \sqrt{\frac{gR_E^{\ 2}}{r}} = R_E\sqrt{\frac{g}{R_E + h}} \qquad [\because r = R_E + h]$$

$$\therefore \boxed{v = R_E\sqrt{\frac{g}{R_E + h}}}$$

- Time period – Time taken by a satellite to complete one revolution around the earth and is denoted by $T$

$$T = \frac{\text{Distance travelled in one revolution}}{\text{Orbital velocity}}$$

$$T = \frac{2\pi r}{v}$$

$$T = \frac{2\pi r}{R_E}\sqrt{\frac{r}{g}}$$

$$T = \frac{2\pi}{R_E}\sqrt{\frac{r^3}{g}} \qquad [\because r = R_E + h]$$

$$\therefore T = \frac{2\pi}{R_E}\sqrt{\frac{(R_E + h)^3}{g}} \qquad \text{...(i)}$$

If the earth is supposed to be a sphere, then mass of the earth is

$$M_E = \frac{4}{3}\pi R_E^3 \rho$$

Where, ρ is the density of earth

$$g = \frac{G\,M_E}{R_E^2} = \frac{G}{R_E^2}\left(\frac{4}{3}\pi R_E^3 \rho\right)$$

$$g = \frac{4}{3}\pi G R_E \rho$$

Substituting this value in equation (i),

$$T = \frac{2\pi}{R_E}\sqrt{\frac{3(R_E + h)^3}{4\pi R_E G\rho}}$$

$$T = \sqrt{\frac{3\pi(R_E + h)^3}{G\rho R_E{}^3}}$$

For a satellite orbiting close to the surface of earth,

$h << R_E$

$\therefore R_E + h \approx R_E$

From equation (1),

$$T = \frac{2\pi}{R_E}\sqrt{\frac{R_E^3}{g}} = 2\pi\sqrt{\frac{R_E}{g}}$$

**Energy of an orbiting satellite**

Total mechanical energy of a satellite revolving around the earth is the sum of its potential energy (*U)* and kinetic energy (*K*).

Potential energy, $U = -\frac{GM_E m}{r}$

Kinetic energy of satellite,

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}\frac{GM_E m}{r}$$

Total energy = $U + K$

$$E = -\frac{GM_E m}{r} + \frac{1}{2}\frac{GM_E m}{r}$$

$$E = -\frac{1}{2}\frac{GM_E m}{r}$$

$$\boxed{E = -\frac{1}{2}\frac{GM_E m}{(R_E + h)}} \qquad [\because r = R_E + h]$$

**Geostationary and Polar Satellites and Weightlessness**

**Geostationary Satellite**

A satellite in a circular orbit around the earth in the equatorial plane, which appears stationary to an observer on the earth, is called geostationary satellite.

- It appears stationary to an observer on the earth.
- The angular speed of the geostationary satellite is synchronised with the angular speed of earth about its axis.
- Conditions for geostationary satellite:
- It should be at a height nearly 36000 km above the equator of earth.
- It should revolve in an orbit concentric and coplanar with the equatorial plane.
- The sense of rotation of the satellite should be same as that of the earth about its own axis.
- The orbital period of the satellite should be the same as that of the earth about its own axis i.e., 24 hrs.
- Height of geostationary satellite

$$h = \left[\frac{gR_E^2T^2}{4\pi^2}\right]^{\frac{1}{3}} - R_E$$

Substituting the values of g = 9.8 $ms^{-2}$,

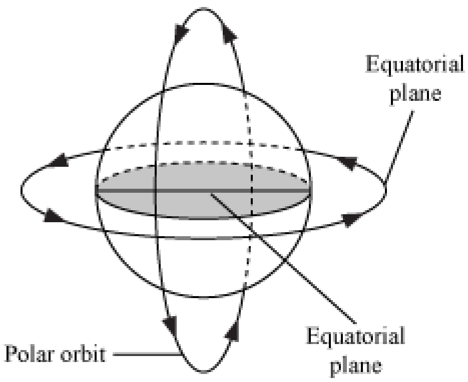
$T$ = 1 day = 86400 s, $R_E$ = 6400 km = $6.4 \times 10^6$ m,

we obtain $h \approx 36000$ km

- Uses – in communicating radio, T.V., and telephone signals across the oceans

**Polar Satellites**

- Polar orbit is that orbit whose angle of inclination with equatorial plane of earth is 90°.




- Polar satellites are low altitude ($h \approx 500 - 800$ km) satellites, which circle the globe in a North-South orbit passing over the North and South poles.

- Polar satellites cross the equator at the same time daily. This is because they are sun synchronous.

- Uses – in taking pictures of clouds; can monitor the climatic changes; useful for remote sensing, meteorology as well as for environmental studies; used for spying and surveillance; also used to monitor the growth of crops

**Parking Orbit**

It is a temporary orbit around the Earth where the satellite is temporarily parked before it is launched to its desired orbit.

**Weightlessness**

- It is the phenomenon in which the observed weight of the body becomes zero.

- Weight of a body is the force with which the body is attracted by the earth towards its centre. [$W$ = mg]

- The reduction in weight is due to the fact that the Earth's gravity grows weaker with increasing altitude.

- An astronaut experiences weightlessness because the earth's gravitational pull on him is used up in providing the centripetal force of rotation around the earth.

- Weightlessness can be achieved only in deep space, very far from any star or planet.

- It can be achieved in the following situations:
- Weight of a body in a freely falling lift is zero.
- Weight of a body is zero at the point where the gravitational fields of earth and moon cancel out. This point is called zero-gravity point.
- At the centre of the earth (where $g = 0$), the weight of a body is zero.